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Gradient Estimates of Periodic Solutions for Some Quasilinear Parabolic Equations

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1. INTRODUCTION

In this paper we are mainly concerned with the estimates of $\|\nabla u(t)\|_\infty$ for periodic solutions of the quasilinear parabolic equation

$$u_t - \operatorname{div}\left\{\sigma\left(|\nabla u|^2\right)\nabla u\right\} + g(x,u) = f(x,t) \qquad \text{in } \Omega \times R \tag{1.1}$$

with the boundary and periodicity conditions

$$u(x,t)_{|\partial\Omega} = 0 \qquad \text{and} \qquad u(x,t+\omega) = u(x,t), \tag{1.2}$$

where Ω is a bounded domain in R^N with a smooth, say C^3-class, boundary $\partial\Omega$, $f(x,t)$ is an ω-periodic (in t) function, $\sigma(v^2)$ is a function like $\sigma(v^2) = |v|^m$, $m \geq 0$, and $g(x,u)$ is a nonlinear function.

To explain our problem let us consider the case $\sigma(|\nabla u|^2) = |\nabla u|^m$. When $g(x,u)u \geq 0$ or more generally when $g(x,u)u \geq -k_0|u|^{\beta+1} - k_1|u|$, $0 \leq \beta < m+1$, and $g(x,u)$ satisfies a little more additional condition, we know that the problem (1.1)–(1.2) admits an ω-periodic solution $u(t)$

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satisfying

$$\int_0^\omega \|u_t(t)\|_2^2\,dt + \sup_t\left\{\|\nabla u(t)\|_{m+2}^{m+2} + \int_\Omega (g(x,u(t))u(t))^+\,dx\right\}$$

$$\leq C\int_0^\omega \|f(t)\|^2\,dt, \tag{1.3}$$

if $f \in L^2(\omega; L^2(\Omega))$ (cf. Seidman [10], Nakao [5]). Hereafter we denote by $C(\omega; X)$, X being a Banach space, the set of continuous ω-periodic functions from R to X. Similar notation will be employed freely. Such a solution as (1.3) is unique if $g(x,u)$ is nondecreasing in u. If we assume further that f belongs to $L^\infty(\omega; L^\infty(\Omega))$, then the solution belongs to $L^\infty(\Omega \times R)$. Indeed, a detailed estimate of $\|u(t)\|_\infty$ is given in [5] when $g(x,u) = 0$.

The object of this paper is to show that the periodic solution belongs further to $L^\infty(\omega; W^{1,\infty}(\Omega))$ and to give a bound of $\|\nabla u(t)\|_\infty$ under a certain geometric assumption on $\partial\Omega$. Of course, we assume in this case $f \in L^\infty(\omega; W^{1,\infty}(\Omega))$.

When $-\operatorname{div}\{\sigma\,\nabla u\}$ is uniformly elliptic and $g(x,u)u \geq \varepsilon_0|u|^2$, $\varepsilon_0 > 0$, the existence of a smooth periodic solution has been discussed by several authors (Smulev [11], Bange [2], etc.). But, for the case like $\sigma = |\nabla u|^m$, $m > 0$, no result is known for the bound of $\|\nabla u(t)\|_\infty$.

For the initial-boundary value problem some gradient estimates have been derived by several authors even for the non-uniformly elliptic case of $-\operatorname{div}\{\sigma\,\nabla u\}$ under the geometric condition on $\partial\Omega$ that the mean curvature $H(x)$ with respect to the outward normal is nonpositive (cf. Engler, Kawohl, and Luckhaus [3], Nakao and Ohara [7], Ohara [8]; see also Alikakos and Rostamian [1] for a related work). In the present paper, we derive, under the same condition on $\partial\Omega$, a bound of $\|\nabla u(t)\|_\infty$ for periodic solutions of the problem (1.1)–(1.2) with σ like $\sigma = |\nabla u|^m$, $m > 0$. We also discuss anti-periodic solutions (cf. H. Okochi [9], A. Haraux [4]). For references on periodic solutions of partial differential equations see O. Vejvoda [12].

2. STATEMENT OF THE RESULT

We use only familiar function spaces and we omit the definition of them.

Our hypotheses on σ, g, f, and $\partial\Omega$ are as follows.

HYPOTHESIS A. *$\sigma(\cdot)$ is differentiable on R^+ and satisfies the conditions*

$$(1) \qquad \sigma(v^2) \geq k_0|v|^m \qquad \text{and} \qquad \sigma'(v^2)v^2 \geq k_0|v|^m \tag{2.1}$$

with some $k_0 > 0$, *and*

$$\text{(2)} \qquad \int_0^{v^2} \sigma(\eta)\, d\eta \geq k_1 \sigma(v^2) v^2 \qquad \text{for } v \in R \tag{2.2}$$

for some $m \geq 0$ *with* $k_1 > 0$.

HYPOTHESIS B. *$g(x, u)$ belongs to $C^1(\Omega \times R)$ and satisfies the conditions*

$$\text{(1)} \qquad g(x, u)u \geq -k_2|u| - k_3|u|^{\beta+1} \tag{2.3}$$

with some $k_2, k_3 > 0$ *and* $0 \leq \beta < m + 1$, *and*

$$\text{(2)} \quad k_4\{g(x, u)u\}^+ \leq \left\{\int_0^u g(x, \eta)\, d\eta\right\}^+ \leq k_5\{g(x, u)u\}^+ \tag{2.4}$$

with some $k_4, k_5 > 0$, *where we use the notation* $\alpha^+ = \max\{\alpha, 0\}$.

($g(x, u) = -|u|^\beta u$ and $g(x, u) = |u|^\alpha u$, $\alpha \geq 0$, are typical examples satisfying Hypothesis B.)

HYPOTHESIS C. *f belongs to $L^\infty(\omega; W_0^{1,\infty}(\Omega))$.*

HYPOTHESIS D. *$\partial\Omega$ is of C^2 class and the mean curvature $H(x)$ at $x \in \partial\Omega$ is nonpositive with respect to the outward normal.*

Our main result is:

THEOREM 1. *Under Hypotheses* A, B, C, *and* D, *there exists an ω-periodic solution $u(t)$ of the problem* (1.1)–(1.2) *in the class*

$$W^{1,2}\big(\omega; L^2(\Omega)\big) \cap L^\infty\big(\omega; W_0^{1,\infty}(\Omega)\big)$$

and the estimates

$$\int_0^\omega \|u_t(t)\|_2^2\, dt + \sup_t \|\nabla u(t)\|_{m+2}^{m+2} \leq C(M_0) < \infty \tag{2.5}$$

and

$$\sup_t \|\nabla u(t)\|_\infty \leq C(M_1) < \infty \tag{2.6}$$

hold for some $C(M_i)$, $i = 0, 1$, depending on M_i, where we set

$$M_0 = \left(\int_0^\omega \|f(t)\|_2^2\, dt\right)^{1/2} \qquad \text{and} \qquad M_1 = \sup_t \|\nabla f(t)\|_\infty.$$

Recently, A. Haraux has proved in [4] the existence of an ω anti-periodic solution $u(t)$ of the semilinear problem, i.e., (1.1)–(1.2) with $\sigma = 1$, under the condition on $g(x,u)$,

$$|g(x,u)| \le k_6\big(|u|^{\beta} + 1\big), \tag{2.7}$$

where $0 < \beta < 1 + 4/N$. Here, we say $u(t)$ is ω anti-periodic if $u(x, t + \omega) = -u(x,t)$ for $t \in R$.

The condition (2.7) is substantially weaker than (2.3) (with $m = 0$). If we restrict ourselves to anti-periodic solutions we can derive a gradient estimate as in Theorem 1 under a weaker assumption than (2.3). That is, we replace Hypotheses B and C by the following ones

HYPOTHESIS B′. *Hypothesis* B *holds with* β *in* (2.3) *replaced by* $0 \le \beta < m + 1 + 2(m+2)/N$.

HYPOTHESIS C′. *f is an ω anti-periodic function belonging to* $L^{\infty}(2\omega; W_0^{1,\infty}(\Omega))$.

THEOREM 2. *Under Hypotheses* A, B′, C′, *and* D *there exists an ω anti-periodic solution $u(t)$ in the class* $W^{1,2}(2\omega; L^2(\Omega)) \cap L^{\infty}(2\omega; W_0^{1,\infty}(\Omega))$ *and the estimates* (2.5) *and* (2.6) *hold*.

We note that in our theorem no growth condition is assumed on $g(x,u)u$, which comes from the L^{∞} estimate of the solutions. We shall briefly discuss L^{∞} estimates in the next section.

Here, we give an elementary lemma concerning a periodic function.

LEMMA 1. *Let $\phi(t) \in C^1(R)$ be a nonnegative ω periodic function satisfying the differential inequality*

$$\phi'(t) + A\phi(t)^{\alpha+1} \le B\phi(t) + C, \qquad t \in R,$$

with some $\alpha > 0$, $A > 0$, $B \ge 0$, *and* $C \ge 0$. *Then, we have*

$$\phi(t) \le \max\Big\{1, \big(A^{-1}(B+C)\big)^{1/\alpha}\Big\}.$$

Proof. If there would exist $t_0 \in [0, \omega]$ such that

$$A\phi(t_0)^{\alpha+1} > B\phi(t_0) + C$$

we see $\phi'(t_0) < 0$. Hence,

$$\phi(t) > \phi(t_0) \qquad \text{for } t \in (t_0 - \delta, t_0)$$

with some δ. But, on this interval we easily see

$$A\phi(t)^{\alpha+1} - B\phi(t) - C > A\phi(t_0)^{\alpha+1} - B\phi(t_0) - C.$$

This implies $\phi'(t) < 0$ on $(-\infty, t_0]$, which contradicts the periodicity of $\phi(t)$. Thus we have

$$A\phi(t)^{\alpha+1} \leq B\phi(t) + C$$

for all t. Letting $x_0 \geq 0$ be the root of the algebraic equation $Ax^{\alpha+1} = Bx + C$, we have from above that $\phi(t) \leq x_0$. But, it is easy to see

$$x_0 \leq \max\left\{1, \left(A^{-1}(B+C)\right)^{1/\alpha}\right\}.$$

3. SOME PRELIMINARY ESTIMATES

In this section we give some preliminary a priori estimates for an assumed smooth periodic solution which are essentially included in [5]. We also give parallel results for an anti-periodic solution under Hypotheses B′ and C′.

Proposition 1. *Let $u(t)$ be a solution of the problem* (1.1)–(1.2) *in the class* $W^{1,2}(\omega; L^2(\Omega)) \cap L^\infty(\omega; W_o^{1,m+2}(\Omega)) \cap L^\infty(\omega; L^\infty(\Omega))$. *Then, under Hypotheses* A, B, *and* C,

$$\int_0^\omega \|u_t(t)\|_2^2\,dt + \|\nabla u(t)\|_{m+2}^{m+2} + \int_\Omega \{g(x, u(t))u(t)\}^+\,dx \leq C(M_0) < \infty. \tag{3.1}$$

Proof. For convenience of the readers we sketch the proof briefly.

Multiplying the equation by u_t and integrating over $[0, \omega] \times \Omega$ we have

$$\int_0^\omega \|u_t(t)\|_2^2\,dt \leq \int_0^\omega \int_\Omega |u_t f|\,dx\,dt,$$

which implies immediately

$$\int_0^\omega \|u_t(t)\|_2^2\,dt \leq M_0^2. \tag{3.2}$$

Next, multiplying the equation by u and integrating we have

$$\int_0^\omega \int_\Omega \sigma |\nabla u|^2 \, dx\, dt + \int_0^\omega \int_\Omega g(x,u) u \, dx\, dt = \int_0^\omega \int_\Omega fu \, dx\, dt$$

and, by the assumptions,

$$\begin{aligned}\int_0^\omega \int_\Omega \sigma |\nabla u|^2 \, dt &+ \int_0^\omega \int_\Omega (g(x,u)u)^+ \, dx\, dt \\ &\le C \int_0^\omega \left(\|u(t)\|_{\beta+1}^{\beta+1} + \|u(t)\|_1 \right) dt \\ &\quad + C \left(\int_0^\omega \|f(t)\|_{m+2}^{(m+2)/(m+1)} \right)^{(m+1)/(m+2)} \left(\int_0^\omega \|u(t)\|_{m+2}^{m+2} \, dt \right)^{1/(m+2)}. \end{aligned} \tag{3.3}$$

Since $\sigma |\nabla u|^2 \ge k_0 |\nabla u|^{m+2}$ and $\beta < m+1$ we have

$$\int_0^\omega \int_\Omega \sigma |\nabla u|^2 \, dx\, dt + \int_0^\omega \int_\Omega \{g(x,u)u\}^+ \, dx\, dt \le C(M_0) < \infty. \tag{3.4}$$

From Hypothesis A(2), Hypothesis B(2), and (3.3) there exists $t^* \in [0, \omega]$ such that

$$\tfrac{1}{2} \int_\Omega \int_0^{|\nabla u(t^*)|^2} \sigma(\eta) \, d\eta \, dx + \int_\Omega \left\{ \int_0^{u(t^*)} g(x,\eta) \, d\eta \right\}^+ dx \le C(M_0). \tag{3.5}$$

Thus, multiplying the equation by $u_t(t)$ and integrating over $[t^*, t] \times \Omega$ or $[t^*, t+\omega] \times \Omega$ we have

$$\begin{aligned} \tfrac{1}{2} \int_\Omega \int_0^{|\nabla u(t)|^2} \sigma(\eta) \, d\eta \, dx &+ \int_\Omega \int_0^{u(t)} g(x,\eta) \, d\eta \, dx \\ &\le \tfrac{1}{2} \int_\Omega \int_0^{|\nabla u(t^*)|^2} \sigma(\eta) \, d\eta \, dx + \int_\Omega \int_0^{u(t^*)} g(x,\eta) \, d\eta \, dx \\ &\quad + \int_0^\omega \int_\Omega |fu_t| \, dx\, ds \\ &\le C(M_0) \qquad \text{for } t \in [0, \omega]. \end{aligned} \tag{3.6}$$

Thus, we obtain (3.1) from (3.2) and (3.6).

PROPOSITION 2. *For an assumed smooth solution of the problem* (1.1)–(1.2) *we have*

$$\sup_t \|u(t)\|_\infty \le C(M_1) < \infty. \tag{3.7}$$

Proof. For convenience of the readers and also for later use we sketch the proof briefly.

Multiplying the equation by $|u(t)|^p u(t)$, $p > 0$, and integrating we have

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2} + \int_\Omega \left\{(p+1)\sigma|u(t)|^p|\nabla u|^2 + g(x,u)|u|^p u\right\} dx$$
$$= \int_\Omega f|u|^p u\, dx$$

and, by the assumptions,

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2} + \varepsilon_0(p+1)\left(\frac{m+2}{p+m+2}\right)^{m+2}\|\nabla\left(|u|^{p/(m+2)}u\right)\|_{m+2}^{m+2}$$
$$\leq C\left(\|u(t)|_{p+\beta+1}^{p+\beta+1} + (M_1+1)\|u(t)\|_{p+2}^{p+1}\right). \tag{3.8}$$

Here,

$$\begin{aligned}\|u(t)\|_{p+\beta+1}^{p+\beta+1} &\leq C\|u(t)\|_{p+2}^{\theta_1}\|u(t)\|_l^{\theta_2}\|u(t)\|_q^{\theta_3}\\ &\leq C\|u(t)\|_{p+2}^{\theta_1}\|\nabla u(t)\|_{m+2}^{\theta_2}\\ &\quad\times\|\nabla\left(|u(t)|^{p/(m+2)}u(t)\right)\|_{m+2}^{\theta_3(m+2)/(p+m+2)},\end{aligned} \tag{3.9}$$

where l, q, θ_i, $i = 1,2,3$, are determined in such a way that

$$l = \begin{cases}(m+2)N/(N-m-2) & \text{if } N > m+2\\ \nu(m+2),\ \nu\text{:large}, & \text{if } N = m+2,\end{cases}$$

$$q = \begin{cases}(p+m+2)N/(N-m-2) & \text{if } N > m+2\\ \nu(p+m+2),\ \nu\text{:large}, & \text{if } N = m+2,\end{cases}$$

$$\theta_1 + \theta_2 + \theta_3 = p+\beta+1, \qquad \frac{\theta_1}{p+2} + \frac{\theta_2}{l} + \frac{\theta_3}{q} = 1, \qquad \text{and}$$

$$\frac{\theta_1}{p+2} + \frac{\theta_3}{p+m+2} = 1,$$

that is,

$$\theta_1 = (p+2)\left\{1 - \frac{(\beta-1)^+ q}{M}\right\}, \qquad \theta_2 = \frac{l(\beta-1)^+(q-p-m-2)}{M}$$

and

$$\theta_3 = \frac{q(\beta-1)^+(p+m+2)}{M}$$

with $M = q(m+l) - l(p+m+2)$.

We require $\theta_3 < p + m + 2$, i.e., $\theta_1 > 0$, which means

$$\beta < \frac{m + 2 + (m+1)N}{(N - m - 2)^+}. \tag{3.10}$$

Since $\beta < m + 1$ the condition (3.10) is certainly satisfied.

It follows from (3.8) and (3.9) that

$$\begin{aligned}
&\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2} + \frac{\varepsilon_0}{2}(p+1)\left(\frac{m+2}{p+m+2}\right)^{m+2}\left\|\nabla\left(|u|^{p/(m+2)}u\right)\right\|_{m+2}^{m+2}\\
&\quad\le C(p+2)^{\alpha}\|\nabla u(t)\|_{m+2}^{\theta_2(p+2)/\theta_1}\|u(t)\|_{p+2}^{p+2} + C(M_1+1)\|_{p+2}^{p+2}\\
&\quad\le C(M_0)(p+2)^{\alpha}\|u(t)\|_{p+2}^{p+2} + C(M_1+1)\|u(t)\|_{p+2}^{p+1}
\end{aligned} \tag{3.11}$$

for a certain $\alpha > 0$ independent of p (note that $\theta_2(p+2)/\theta_1 < (\beta - 1)^+$).

Now, taking $p_0 = m$ and $p_n = (m+2)p_{n-1} + m + 2$ we have, by a variant of the Gagliardo–Nirenberg inequality,

$$\|u(t)\|_{p_n+2} \le C\|u(t)\|_{p_{n-1}+2}^{1-\delta_n}\left\|\nabla\left(|u|^{p_n/(m+2)}u\right)\right\|_{m+2}^{\delta_n(m+2)/(p_n+m+2)} \tag{3.12}$$

with

$$\delta_n = \frac{N\left((m+2)p_n + m + 2\right)}{\left((m+1)N + m + 2\right)(p_n + 2)}.$$

We have from (3.11) and (3.12) that

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{p_n+2} + \varepsilon_1(p_n+1)\left(\frac{m+2}{p_n+m+2}\right)^{m+2}\\
&\qquad\times\|u(t)\|_{p_{n-1}+2}^{-(p_n+m+2)(1-\delta_n)/\delta_n}\|u(t)\|_{p_n+2}^{(p_n+m+2)/\delta_n - p_n - 1}\\
&\qquad\le C(M_1)\left(p_n^{\alpha}\|u(t)\|_{p_n+2} + 1\right)
\end{aligned} \tag{3.13}$$

for some $\varepsilon_1 > 0$.

Setting $\chi_n \equiv \sup_t \|u(t)\|_{p_n+2}$, (3.13) is rewritten as

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{p_n+2} + \varepsilon_1(p_n+1)\left(\frac{m+2}{p_n+m+2}\right)^{m+2}\chi_{n-1}^{m-\beta_n}\|u(t)\|_{p_n+2}^{\beta_n+1}\\
&\qquad\le C(M_1)p_n^{\alpha}\left(\|u(t)\|_{p_n+2} + 1\right)
\end{aligned} \tag{3.14}$$

with

$$\beta_n = \frac{(p_n + 2)(m + 2)(mN + p_n + m + 2)}{N((m + 1)p_n + m + 2)} > m.$$

The differential inequality (3.14) implies, by the periodicity,

$$\chi_n \leq \max\left\{1, \left(\frac{2C(M_1)p_n^{\alpha}}{\varepsilon_1(p_n + 1)}\left(\frac{p_n + m + 2}{m + 2}\right)^{m+2}\chi_{n-1}^{\beta_n - m}\right)^{1/\beta_n}\right\}. \quad (3.15)$$

From this we can show that $\{\chi_n\}$ is bounded by a certain $C(M_1) < \infty$ and conclude that

$$\sup_t \|u(t)\|_\infty \leq \varlimsup_{n\to\infty} \chi_n \leq C(M_1) < \infty.$$

For anti-periodic solutions, Propositions 1, 2 hold under Hypotheses A, B′, and C′. Indeed, we replace ω by 2ω to get (3.2) for anti-periodic solutions. Then, by antiperiodicity, we have

$$\sup_t \|u(t)\|_2 \leq CM_0.$$

When $\beta \geq m + 1$ we can treat the term $\|u(t)\|_{\beta+1}$ in (3.3) by using the Gagliardo−Nirenberg inequality as

$$\int_0^\omega \|u(t)\|_{\beta+1}^{\beta+1}\,dt \leq \int_0^\omega \|u(t)\|_2^{(\beta+1)(1-\theta)}\|\nabla u(t)\|_{m+2}^{(\beta+1)\theta}\,dt$$

$$\leq CM_0^{(\beta+1)(1-\theta)}\int_0^\omega \|\nabla u(t)\|_{m+2}^{(\beta+1)\theta}\,dt$$

with $\theta = (1/2 - 1/(\beta + 1))(1/N + 1/2 - 1/(m + 2))^{-1}$, Here, by Hypothesis B′ we easily see $0 < \theta < 1$ and further $(\beta + 1)\theta < m + 2$. Thus, we obtain (3.4) for anti-periodic solutions under Hypothesis B′. For other arguments we have only to replace ω by 2ω. To see the estimate (3.7) we have only to check the condition (3.10). But, this is equivalent to $\theta < 1$ for the above θ and is certainly satisfied under Hypothesis B′. From these a priori estimates we obtain, by use of a standard monotonicity argument, the following existence theorem of an anti-periodic solution (cf. [5]).

PROPOSITION 3. *In addition to Hypotheses* A, B′, C′ *we assume that*

$$\sigma(|\nabla u|^2) \leq k_6(|\nabla u|^m + 1),$$

with some $k_6 > 0$. *Then, there exists an* ω *anti-periodic solution* $u(t)$ *in the class*

$$L^2\big(2\omega; L^2(\Omega)\big) \cap L^\infty\big(2\omega; W_0^{1,m+2}(\Omega)\big) \cap L^\infty(2\omega; L^\infty(\Omega)),$$

satisfying the estimates (3.1) *and* (3.7).

The above result is more general and sharper than the existence theorem for a semilinear equation given in Haraux [4].

4. A PRIORI ESTIMATES FOR $\nabla u(t)$

In this section we derive a priori estimates of $\|\nabla u(t)\|_\infty$ for an assumed smooth solution of the problem (1.1)–(1.2). "Smooth" means that the solution $u(t)$ belongs to $W^{1,2}(\omega; L^2(\Omega)) \cap L^\infty(\omega; W_0^{1,\infty}(\Omega))$. This estimation is the heart of this paper.

PROPOSITION 4. *Under Hypothesis* A, B, C, *and* D, *a* (*smooth*) *periodic solution* $u(t)$ *of the problem* (1.1)–(1.2) *satisfies*

$$\|\nabla u(t)\|_\infty \le C(M_1) < \infty. \tag{4.1}$$

Proof. Multiplying the equation by $-(|\nabla u|^p u_j)_j$ and integrating over Ω we have

$$\begin{aligned}\frac{1}{p+2}\frac{d}{dt}\|\nabla u(t)\|_{p+2}^{p+2} &+ \int_\Omega \operatorname{div}\{\sigma \nabla u\}\big(|\nabla u|^p u_j\big)_j\,dx \\ &= \int_\Omega \big(g_{x_j}(x,u) + g_u u_j + f_j\big)|\nabla u|^p u_j\,dx.\end{aligned} \tag{4.2}$$

Here, integrating by parts, we see (cf. [3, 6])

$$\begin{aligned}&\int_\Omega \operatorname{div}\{\sigma \nabla u\}\big(|\nabla u|^p u_j\big)_j\,dx \\ &\quad = \int_\Omega (\sigma u_i)_j\big(|\nabla u|^p u_j\big)_i\,dx + \int_{\partial\Omega} |\nabla u|^p\big\{(\sigma u_i)_i u_j n_j - (\sigma u_i)_j u_j n_i\big\}\,d\Gamma\end{aligned}$$

$$= \int_\Omega \{2\sigma' u_k u_{kj} u_i + \sigma u_{ij}\}\Big\{|\nabla u|^p u_{ij} + p|\nabla u|^{p-2} u_j u_l u_{li}\Big\}\,dx$$

$$+ \int_{\partial\Omega} |\nabla u|^p \{\sigma u_{ii} u_j n_j + 2\sigma' u_k u_{ki} u_i u_j n_j$$

$$- \sigma u_{ij} u_j n_i - 2\sigma' u_k u_{kj} u_i u_j n_i\}\,d\Gamma$$

$$= \int_\Omega |\nabla u|^p \left\{\sigma \sum_{i,j} (u_{ij})^2 + 2\sigma' \sum_j \left(\sum_i u_i u_{ij}\right)^2\right\} dx$$

$$+ \int_\Omega p|\nabla u|^p \left\{2\sigma' \left(\sum_{k,j} u_k u_j u_{kj}\right)^2 + \sigma \sum_i \left(\sum_j u_j u_{ij}\right)^2\right\} dx$$

$$+ \int_{\partial\Omega} \sigma |\nabla u|^p \frac{\partial u}{\partial n}\left(\nabla u - \frac{\partial^2 u}{\partial n^2}\right) dx$$

$$\geq \frac{\varepsilon_0(p+1)}{4} \int_\Omega |\nabla u|^{p+m-2} |\nabla(|\nabla u|^2)|^2\,dx$$

$$- (N-1) \int_{\partial\Omega} \sigma |\nabla u|^{p+2} H(x)\,d\Gamma \tag{4.3}$$

$$\geq \frac{4\varepsilon_0(p+1)}{(p+m+2)^2} \int_\Omega |\nabla(|\nabla u|^{(p+m+2)/2})|^2\,dx \tag{4.4}$$

for some $\varepsilon_0 > 0$, where we have used the assumption $H(x) \leq 0$ at the last stage.

Further we see

$$\int_\Omega \Big|\Big(g_{x_j}(x,u) + g_u(x,u) u_j\Big)\Big| |\nabla u|^p u_j\,dx$$

$$\leq C(M_1)\Big(\|\nabla u(t)\|_{p+2}^{p+2} + \|\nabla u(t)\|_{p+2}^{p+1}\Big) \tag{4.5}$$

and

$$\int_\Omega |f_j(x,t)|\,|\nabla u|^p |u_j|\,dx \leq C\|\nabla f\|_{p+2} \|\nabla u\|_{p+2}^{p+1} \leq CM_2 \|\nabla u\|_{p+2}^{p+1}. \tag{4.6}$$

It follows from (4.2)–(4.5) and (4.6) that

$$\frac{1}{p+2}\frac{d}{dt}\|\nabla u(t)\|_{p+2}^{p+2} + \frac{4\varepsilon_0(p+1)}{(p+m+2)^2}\Big\|\nabla\big(|\nabla u|^{(p+m+2)/2}\big)\Big\|_2^2$$

$$\leq C(M_2)\Big(\|\nabla u(t)\|_{p+2}^{p+2} + \|\nabla u(t)\|_{p+2}^{p+1}\Big). \tag{4.7}$$

The procedure to derive a bound of $\|\nabla u(t)\|_\infty$ from (4.7) is very similar to the one obtaining (3.7). Indeed, we first note, by a variant of the Gagliardo–Nirenberg inequality and Hölder's inequality (cf. [7, 8]),

$$2\|\nabla u(t)\|_{p+m+2}^{p+m+2} \le \left\| |\nabla u(t)|^{(p+m+2)/2} \right\|_{W^{1,2}}^2 + C\|\nabla u(t)\|_{m+2}^{m+1}\|\nabla u(t)\|_{p+2}^{p+1}$$

and hence,

$$\left\| |\nabla u(t)|^{(p+m+2)/2} \right\|_{W^{1,2}}^2 = \left\| \nabla\left(|\nabla u(t)|^{(p+m+2)/2}\right) \right\|_2^2 + \|\nabla u(t)\|_{p+m+2}^{p+m+2}$$
$$\le 2\left\| \nabla\left(|\nabla u|^{(p+m+2)/2}\right) \right\|_2^2 + C(M_0)\|\nabla u(t)\|_{p+2}^{p+1}.$$

Therefore, (4.7) can be rewritten as

$$\frac{1}{p+2}\frac{d}{dt}\|\nabla u(t)\|_{p+2}^{p+2} + \frac{2\varepsilon_0(p+1)}{(p+m+2)^2}\left\| |\nabla u(t)|^{(p+m+2)/2} \right\|_{W^{1,2}}^2$$
$$\le C(M_2)\left(\|\nabla u(t)\|_{p+2}^{p+2} + \|\nabla u(t)\|_{p+2}^{p+1}\right). \tag{4.8}$$

Then, setting

$$p_0 = m \qquad \text{and} \qquad p_n = 2p_{n-1} - m + 2, \qquad n = 1, 2, \dots,$$

we have, by a variant of the Gagliardo–Nirenberg inequality, again,

$$\|\nabla u(t)\|_{p_n+2}^{p_n+2} \le C\|\nabla u(t)\|_{p_{n-n}+2}^{(1-\theta_n)(p_n+2)}\left\| |\nabla u(t)|^{(p_n+m+2)/2} \right\|_{W^{1,2}}^{2(p_n+2)\theta_n/(p_n+m+2)} \tag{4.9}$$

with

$$\theta_n = \frac{p_n+m+2}{2}\left(\frac{1}{p_{n-1}+2} - \frac{1}{p_n+2}\right)$$
$$\times\left(\frac{1}{N} - \frac{1}{2} + \frac{1}{p_{n-1}+2}\cdot\frac{p_n+m+2}{2}\right)^{-1}$$
$$= (p_n+2-m)N/(p_n+2)(N+2). \tag{4.10}$$

Therefore, from (4.8) and (4.9),

$$\frac{d}{dt}\|\nabla u(t)\|_{p_n+2} + \varepsilon_1 p_n^{-1}\|\nabla u(t)\|_{p_{n-1}+2}^{m-\beta_n}\|\nabla u(t)\|_{p_n+2}^{\beta_n+1}$$
$$\le C(M_2)\left(\|\nabla u(t)\|_{p_n+2} + 1\right), \tag{4.11}$$

where we set, for convenience,

$$\beta_n = (p_n + m + 2)/\theta_n - p_n - 2. \tag{4.12}$$

Setting $\chi_n = \sup_t \|\nabla u(t)\|_{p_n+2}$, the differential inequality (4.11) implies, by the periodicity (cf. [5]),

$$\chi_n \leq \max\left\{1, \left(2\varepsilon_1^{-1} C(M_2) p_n \chi_{n-1}^{\beta_n - m}\right)^{1/\beta_n}\right\}. \tag{4.13}$$

If $\chi_{n'} \leq 1$ for a subsequence we get immediately $\|\nabla u(t)\|_\infty \leq 1$. So, we may assume $\chi_n \geq 1$. Then, setting $\omega_n = \log \chi_n$, we have

$$\begin{aligned} \omega_n &\leq \frac{1}{\beta_n}\{(\beta_n - m)\omega_{n-1} + \log C(M_2) + \log p_n\} \\ &\leq \omega_{n-1} + 2^{-n}\{C(M_2) + Cn\} \end{aligned} \tag{4.14}$$

for some constants $C(M_2)$ and C. Inequality (4.14) implies easily

$$\omega_n \leq \omega_0 + C(M_2) < \infty,$$

which immediately gives, by taking $n \to \infty$ along a subsequence,

$$\|\nabla u(t)\|_\infty \leq C_2(M_2) \equiv \chi_0 e^{C(M_2)} < \infty.$$

Finally, we note that the above argument can be applied to ω anti-periodic solutions under Hypotheses B′ and C′ for B and C, because we know already $\|u(t)\|_\infty \leq C(M_1) < \infty$ for anti-periodic solutions under the hypotheses.

5. PROOF OF THEOREMS

On the basis of the a priori estimates in the previous section we give a proof of Theorem 1. Theorem 2 can be treated quite similarly.

We first approximate $\sigma(v^2)$ by a smooth function $\sigma_\varepsilon(v^2)$, $\varepsilon > 0$, such that

$$\sigma_\varepsilon(v^2) \geq k_0|v|^m + \varepsilon \qquad \text{and} \qquad \sigma_\varepsilon'(v^2)v^2 \geq k_0|v|^m \tag{5.1}$$

and

$$\sigma_\varepsilon(v^2) \to \sigma(v^2) \qquad \text{uniformly on compact subsets of } R \tag{5.2}$$

as $\varepsilon \to 0$.

Next, we approximate $\sigma_\varepsilon(v^2)$ by a smooth function $\sigma_{\varepsilon, L}(v^2)$, $L > 0$, such that

$$\sigma_{\varepsilon, L}(v^2) = \sigma_\varepsilon(v^2) \qquad \text{if } |v| \leq L, \tag{5.3}$$

$$\sigma_{\varepsilon, L}(v^2) = \max_{|v| \leq L} \sigma_\varepsilon(v^2) + 1 \qquad \text{if } |v| \geq L + 1, \tag{5.4}$$

and

$$\sigma'_{\varepsilon, L}(v^2) \geq 0. \tag{5.5}$$

Further, we approximate $g(x, u)$ and $f(x, t)$ by smooth functions $g_\varepsilon(x, u)$ and $f_\varepsilon(x, t)$, respectively, such that

$$g_\varepsilon(x, u)u \geq -\frac{k_0}{2}|u|^{\beta+1} - \frac{k_1}{2}|u| + \varepsilon|u|^{\beta+2},$$

$$g_\varepsilon(x, u) \to g(x, u) \qquad \text{uniformly on compact subsets of } \overline{\Omega} \times R \text{ as } \varepsilon \to 0$$

and

$$f_\varepsilon(x, t) \to f(x, t) \qquad \text{uniformly as } \varepsilon \to 0.$$

Let us consider the approximate problem

$$\begin{cases} u_t - \operatorname{div}\left\{\sigma_{\varepsilon, L}\left(|\nabla u|^2\right) \nabla u\right\} + g_\varepsilon(x, u) = f_\varepsilon(x, t) \\ u_{|\partial\Omega} = 0, \qquad u(x, t) = u(x, t + \omega) \ (\text{or } u(x, t + \omega) = -u(x, t)). \end{cases} \tag{$P_{\varepsilon, L}$}$$

Since our operator $-\operatorname{div}\{\sigma_{\varepsilon, L} \nabla u\}$ is uniformly elliptic, a standard argument based on Leray–Shauder degree theory (cf. Smulev [11]) the problem $(P_{\varepsilon, L})$ admits a smooth ω-periodic (or anti-periodic) solution $u_{\varepsilon, L}(t)$.

Applying the a priori estimates in the previous section (with $m = 0$) we see that

$$\sup_t \|\nabla u_{\varepsilon, L}(t)\|_\infty \leq C(\varepsilon) < \infty. \tag{5.6}$$

Taking $L > C(\varepsilon)$, we find that $u_{\varepsilon, L}(t)$ is in fact a smooth solution of the problem

$$\begin{cases} u_t - \operatorname{div}\left\{\sigma_\varepsilon\left(|\nabla u|^2\right) \nabla u\right\} + g_\varepsilon(x, u) = f_\varepsilon(x, t) \\ u_{|\partial\Omega} = 0, \qquad u(x, t) = u(x, t + \omega) \ (\text{or } u(x, t + \omega) = -u(x, t)). \end{cases} \tag{P_ε}$$

It is clear that the estimates in the previous section can be applied to $u_\varepsilon(t) \equiv u_{\varepsilon, L}(t)$ and all the bounds can be taken to be independent of ε. Thus, taking a limit of $u_\varepsilon(t)$ as $\varepsilon \to 0$, along a subsequence, we know that $u_\varepsilon(t)$ converges to a function $u(t)$ in such a way that

$$u_\varepsilon(t) \to u(t) \qquad \text{weakly* in } L^\infty\big(\omega; W_0^{1,\infty}(\Omega)\big),$$
$$u_{\varepsilon t}(t) \to u_t(t) \qquad \text{weakly in } L^2\big(\omega; L^2(\Omega)\big),$$

and

$$\sigma_\varepsilon\big(|\nabla u_\varepsilon|^2\big)\nabla u_\varepsilon \to \chi \qquad \text{weakly in } L^{(p+2)/(p+1)}\big(\omega; L^{(p+2)/(p+1)}(\Omega)\big) \quad \text{for } \forall p > 0.$$

Further, we easily see that $W^{1,2}(\omega; L^2(\Omega)) \cap L^\infty(\omega; W_0^{1,\infty})$ is compactly imbedded in $C(\omega; C(\overline{\Omega}))$ and hence, we have

$$u_\varepsilon(t) \to u(t) \qquad \text{in } C\big(\omega; C(\overline{\Omega})\big).$$

(For the proof of Theorem 2 ω should be replaced by 2ω.)

Since $A_\varepsilon(u) \equiv -\operatorname{div}\{\sigma_\varepsilon(|\nabla u|)\nabla u\}$ is a monotone operator from $W_0^{1,\infty}$ to $W^{-1,q}$, $1 < \forall q < 2$, we conclude, by Minty's trick, that $\chi = \sigma(|\nabla u|^2)\nabla u$. Thus, $u(t)$ is a desired solution in Theorem 1 (or Theorem 2). Needless to say, all the estimates for $\{u_\varepsilon\}$ remain valid for $u(t)$, that is, the estimates in the previous section hold for $u(t)$.

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